Problem Set 3 (Due on Friday, 10/29)

Problem 1 *Let $D(n)$ be the number of derangements in S_n.*

(1) Prove that $D(n) = (n-1)(D(n-1) + D(n-2))$.

(2) Deduce that $D(n) = nD(n-1) + (-1)^n$.

Problem 2 *For each $n \in \mathbb{N}_0$, let C_n be the n-th Catalan number and set $a_n = nC_n$. Find an explicit formula for the generating function of $(a_n)_{n\geq 0}$.*

Problem 3 *Find an explicit formula for the number of solutions $(x, y, z) \in \mathbb{N}_0^3$ of the equation $x + y + z = n$ satisfying that x is odd, $y > 2$, and $z < 5$.*

Problem 4 *Let a_n be the number of compositions of n with an odd number of parts such that every part is at least 3. Find an explicit formula (no summation signs allowed) for the generating function of $(a_n)_{n\geq 0}$.*

Problem 5 *Let t_n be the number of partitions of $[n]$ into blocks of cardinality two. Find the explicit formula (no summation signs allowed) for the exponential generating function of $(t_n)_{n\geq 0}$.*

Problem 6 *Find an explicit formula (no summation signs allowed) for the exponential generating function of $(D(n))_{n\geq 0}$, where $D(0) = 1$ and $D(n)$ is the number of derangements of S_n.*

Problem 7 *For each $n \in \mathbb{N}$, let t_n be the number of simple graphs with vertex set $[n]$ with no vertex of degree larger than 2, and assume that $t_0 = 1$. Find an explicit formula for the exponential generating function of $(t_n)_{n\geq 0}$.*

Problem 8 *Using generating functions, prove that the number of partitions of n into distinct parts equals the number of partitions of n where each part is odd.*

Solution. Let $q(n)$ be the number of partitions of n into distinct parts, and let $p_o(n)$ be the number of partitions of n whose parts are odd. Then we see that

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{i=1}^{\infty}(1 + x^i) = \prod_{i=1}^{\infty} \frac{1 - x^{2i}}{1 - x^i} = \frac{\prod_{i=1}^{\infty}(1 - x^{2i})}{\prod_{i=1}^{\infty}(1 - x^i)} = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}}. \tag{0.1}$$

Now, observe that the right-most part of (0.1) is the generating function of the sequence $(p_o(n))_{n\geq 0}$. Thus,

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}} = \sum_{n=0}^{\infty} p_o(x)x^n.$$

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